Journal of Computational Physics **170,** 849–870 (2001)
doi:10.1006/jcph.2001.6766, available online at http://www.idealibrary.com on IDEAL®

High-Order Nonreflecting Boundary Conditions without High-Order Derivatives

Dan Givoli

Department of Aerospace Engineering and Asher Center for Space Research, Technion–Israel Institute of Technology, Haifa 32000, Israel
E-mail: givolid@aerodyne.technion.ac.il

Received August 18, 2000; revised March 2, 2001

A wave problem in an unbounded domain is often treated numerically by truncating the infinite domain via an artificial boundary $\mathcal{B}$, imposing a so-called nonreflecting boundary condition (NRBC) on $\mathcal{B}$, and then solving the problem numerically in the finite domain bounded by $\mathcal{B}$. A general approach is devised here to construct high-order local NRBCs with a symmetric structure and with only low (first- or second-) order spatial and/or temporal derivatives. This enables the practical use of NRBCs of arbitrarily high order. In the case of time-harmonic waves with finite element discretization, the approach yields a symmetric C^0 finite element formulation in which standard elements can be employed. The general methodology is presented for both the time-harmonic case (Helmholtz equation) and the time-dependent case (the wave equation) and is demonstrated numerically in the former case.

Key Words: waves; high-order; artificial boundary; nonreflecting boundary condition; finite element.

1. INTRODUCTION

A common method used for the numerical solution of wave problems in an unbounded domain [1] is based on truncating the domain via an artificial boundary $\mathcal{B}$, thus forming a finite computational domain Ω bounded by $\mathcal{B}$. A so-called nonreflecting boundary condition (NRBC) is imposed on $\mathcal{B}$ to complete the statement of the problem (i.e., to make the solution unique) and, most importantly, to ensure that no (or little) spurious wave reflection occurs from $\mathcal{B}$. Then the problem is solved numerically in Ω (see Fig. 1). Naturally, the quality of the numerical solution strongly depends on the properties of the NRBC employed. In the past two decades, much research has been done to develop NRBCs that after discretization lead to a scheme which is stable, accurate, efficient, and easy to implement; see [2] and [3] for recent reviews on the subject. Of course, it is difficult to find a single NRBC which is ideal in all respects and all cases; this is why the quest for better NRBCs and their associated discretization schemes continues.

0021-9991/01 $35.00

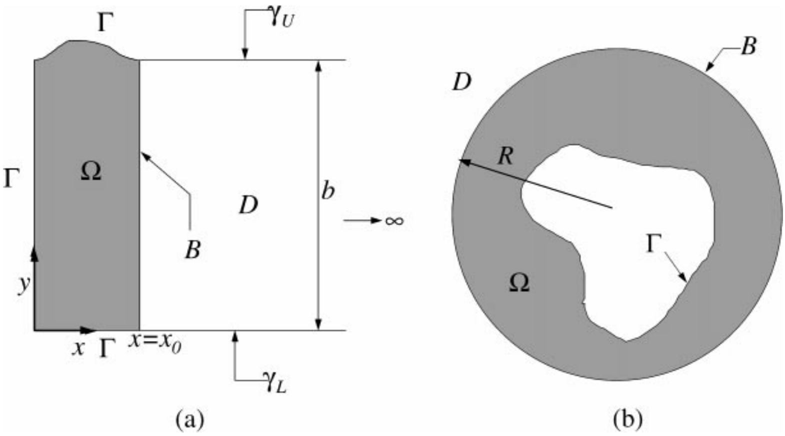

FIG. 1. Setup for the method of NRBCs. The artificial boundary $\mathcal{B}$ truncates the infinite domain, thus forming a finite computational domain Ω. A nonreflecting boundary condition (NRBC) is applied on $\mathcal{B}$, and the problem in Ω is solved numerically. (a) Wave guide configuration. (b) Exterior problem configuration.

The collection of NRBCs that have been proposed can be divided into two sets: *nonlocal* and *local* NRBCs. Nonlocal NRBCs, the main example of which is the Dirichlet-to-Neumann (DtN) condition [1–3], are typically exact (or exact up to a series truncation error which can be made vanishingly small), whereas local NRBCs are usually approximate. Various sequences of local NRBCs with increasing order of accuracy have been devised, e.g., the sequences of Engquist and Majda [4], Bayliss and Turkel [5], Feng [6], Higdon [7], and Givoli *et al.* [8, 9]. The Engquist–Majda NRBCs are based on approximating the symbol of the governing pseudo-differential operator by rational functions, the Bayliss–Turkel and Feng NRBCs are two types of asymptotic conditions, the Higdon NRBCs annihilate wave reflection in a finite number of specified directions, and the NRBCs of Givoli *et al.* are obtained by localizing the exact nonlocal DtN condition in some "optimal" way.

A detailed comparison of the properties of nonlocal and local NRBCs appears in [3]. The nonlocal DtN condition is robust, very accurate, stable, and, in the context of the finite element (FE) method, leads to a well-conditioned symmetric matrix problem. However, it may involve special functions that are not convenient to compute, it must be used on an artificial boundary $\mathcal{B}$ of a simple smooth shape, and it often affects negatively the sparseness of the coefficient matrix in the discrete scheme (e.g., the FE stiffness matrix). It should be noted, however, that in the elliptic case some strong arguments may be made to show that these relative disadvantages may be overcome or that they are not really as negative as one may think [3, 10]. A more serious difficulty is that in some important cases an expression for the exact DtN condition is not available or is too complicated to be practical.

Local NRBCs can be used in principle on a generally shaped boundary $\mathcal{B}$ and are compatible with local FE architecture (where all operations are performed on the element level). However, *low*-order local NRBCs may have low accuracy (at least in some cases, which always exist), whereas *high*-order local NRBCs are usually hard to implement because they *typically involve high-order derivatives*. Clearly, standard C^0 FEs cannot be used in the presence of high-order *tangential* derivatives. Even if the NRBC involves only high *normal* derivatives, this requires the use of high-order FE interpolation, with associated difficulties. The appearance of high-order *temporal* derivatives is also problematic from a numerical time-integration standpoint. In addition, most high-order NRBCs, if used in a straightforward manner, lead to a nonsymmetric and ill-conditioned FE scheme. Similar difficulties exist with other discretization methods, such as finite differences. Until recently a *very-high*-order local NRBC has never been used because of these reasons. The most popular NRBCs are first- and second-order conditions, which are easy to implement but not always sufficiently accurate. Rarely has an NRBC involving a third- or a fourth-order derivative been used.

One approach in the context of FEs to deal with high-order tangential derivatives that occur in an NRBC is to develop and use special FEs that possess high-order regularity on $\mathcal{B}$. A hierarchy of such two-dimensional elements was first introduced in [11], and then extended in [12] to three dimensions and in [8] to p-type elements. The latter paper includes calculations done with a local NRBC involving a sixth-order tangential derivative. While it is possible in principle to use this approach with a very high-order NRBC, this is not so practical since the coding of the special elements is difficult, and moreover each element in the hierarchy must be coded separately.

Recently, some new sequences of local high-order NRBCs that *do not involve high-order derivatives* at all have been proposed by several researchers. Instead, these NRBCs involve *auxiliary variables* defined on $\mathcal{B}$. Thus, the differential order of the NRBC is reduced at the price of increasing the number of unknowns in the problem. If the construction is explicit enough, it can be coded once and for all and thus allows the practical use of an NRBC of an *arbitrarily* high order. We therefore call such NRBC an arbitrarily high-order condition (AHOC).

The first AHOC was apparently devised by Collino [13] for two-dimensional time-dependent waves in rectangular domains. Its construction requires the solution of the one-dimensional wave equation on $\mathcal{B}$. Grote and Keller [14, 15] proposed an AHOC for the three-dimensional time-dependent wave equation based on spherical harmonic transformations. They extended this AHOC for the case of elastic waves in [16]. Hagstrom and Hariharan [17, 18] constructed an AHOC for the two- and three-dimensional time-dependent wave equations based on the analytic series representation for the outgoing solutions of these equations. It looks simpler than the previous two AHOCs. For time-dependent waves in a two-dimensional wave guide, Guddati and Tassoulas [19, 20] devised an AHOC by rewriting the Engquist–Majda sequence and the Higdon sequence of NRBCs as a recursive continued fraction.

We comment on the *exactness* of AHOCs. One measure, on the continuous level, of how accurately a given NRBC solves a specific problem is as follows. One considers the exact solution of two problems: the first is the original problem in the infinite domain, and the second is the problem in the truncated domain Ω, with the given NRBC applied on $\mathcal{B}$. Thus the distance δ, in some reasonable norm, between the two solutions in Ω may serve as an error measure. Now, if an AHOC of order K has the property that its error measure δ approaches zero as K goes to infinity while $\mathcal{B}$ is held fixed, and if this property holds for *any* given wave problem of a class under consideration, then it seems justified to call the AHOC *exact*. In this sense, the Grote–Keller AHOC, the three-dimensional Hagstrom–Hariharan AHOC, and the AHOC based on the localized DtN conditions (see Section 6) are all exact. Note that according to our definition, the convergence property of the NRBC as $K \to \infty$ is not sufficient to merit the title "exact"; the NRBC must also be an AHOC—that is, be implementable for an arbitrarily large K.

In the present paper we present a systematic way to reformulate any given sequence of NRBCs as an AHOC. In other words, for a given sequence of NRBCs involving spatial and/or temporal derivatives of increasing order, we construct another sequence of equivalent NRBCs with only low (first- or second-) order derivatives. We do this both in the time-harmonic case (Helmholtz equation) and in the time-dependent case (wave equation). Interestingly, the time-harmonic case has not been treated in this context so far. Thus, this is the first AHOC proposed for the Helmholtz equation.

One important property that the proposed AHOC possesses is that it has a *symmetric* structure. Thus, if the original problem is self-adjoint, the truncated problem with the proposed AHOC on $\mathcal{B}$ is also self-adjoint and hence leads to a symmetric problem on the discrete level. In the elliptic (time-harmonic) case, there is a *unique* way to obtain such a symmetric AHOC. Therefore we shall call it *the symmetric AHOC*. Considering FE discretization, this symmetry is important for three reasons:

1. It is theoretically satisfying that the original self-adjoint problem (in an unbounded domain) is replaced by a self-adjoint problem (in a finite domain). This would not be the case without the symmetry property of the AHOC.
2. On the discrete level, this property ensures that the NRBC does not spoil the symmetry of the FE formulation and thus allows the use of standard symmetric solvers, which are more efficient than nonsymmetric solvers in computer storage and number of floating-point operations.
3. As will be demonstrated numerically later, the symmetric AHOC is also more stable than the original NRBC.

For the case of the Helmholtz equation, the symmetric AHOC leads to a symmetric C^0 FE formulation, which enables the use of standard FEs. For the time-dependent case we construct an analogous AHOC, which leads, after discretization in space, to a symmetric system of linear ordinary differential equations of a standard form.

Following is the outline of the paper. In Section 2 we show how to construct the symmetric AHOC for the Helmholtz equation, given an initial sequence of NRBCs of increasing order. We discuss various forms of the initial sequence, involving either tangential derivatives or normal derivatives or both. In Section 3 we discuss the FE formulation for the new problem in Ω involving the symmetric AHOC applied on $\mathcal{B}$. In Section 4 we extend the construction of the symmetric AHOC to the time-dependent case, and in Section 5 we discuss the corresponding FE semidiscrete formulation. All this is done for a general initial sequence of NRBCs. In Section 6 we present the results of some numerical experiments for the time-harmonic case using the localized DtN conditions [8, 9] as the initial sequence. We draw conclusions regarding the effectiveness of the symmetric AHOC. We conclude with some remarks in Section 7.

2. SYMMETRIC ARBITRARILY HIGH-ORDER CONDITIONS FOR THE HELMHOLTZ EQUATION

We shall present the proposed methodology in *two dimensions* and in *polar coordinates*. The configuration is illustrated in Fig. 1b. However, other cases, such as three-dimensional and wave-guide configurations, may be treated similarly.

We consider the Helmholtz equation in the plane outside of an obstacle:

$$\nabla^2 u + \kappa^2 u = 0. \tag{1}$$

Here κ is the wavenumber. Some boundary condition is given on the obstacle boundary Γ; to fix ideas we consider the Neumann condition

$$\frac{\partial u}{\partial \nu} = h \quad \text{on } \Gamma, \tag{2}$$

where $\partial u/\partial \nu$ is the normal derivative of u on Γ and h is a given function. At infinity, the Sommerfeld radiation condition holds, which dictates that waves are outgoing there. We introduce a circular artificial boundary $\mathcal{B}$ of radius R, which encloses a finite computational domain Ω (see Fig. 1b). We use the polar coordinate system (r, θ) whose origin is located such that $\mathcal{B}$ is the circle $r = R$.

On $\mathcal{B}$, an NRBC is applied, as discussed in the Introduction. All sequences of local NRBCs mentioned previously have the form (or can be written in the form)

$$-\frac{\partial u}{\partial r} = L_K u \quad \text{on } \mathcal{B}, \tag{3}$$

where L_K is a differential operator, and the index $K = 0, 1, 2, \ldots$ is the order of the NRBC. This form is compatible with FE formulations, since variationally (3) can be treated as a natural boundary condition. We concentrate here on NRBCs of this form.

We begin with the case where L_K involves only *tangential* ($\theta-$) derivatives of *even* order. We consider the case where *odd* orders appear as well. Then we turn to the case where L_K involves only *radial* derivatives. Finally, we address the general case where *both tangential and radial* derivatives appear in L_K.

2.1. *NRBCs Involving Even-Order Tangential Derivatives*

We consider NRBCs that involve only tangential derivatives of even order. Such NRBCs have the form

$$-\frac{\partial u}{\partial r} = \sum_{j=0}^{K} \alpha_j \partial_\theta^{2j} u \quad \text{on } \mathcal{B}, \tag{4}$$

where the α_j are given complex constants (that usually depend on K and R). The NRBC (4) is particularly attractive because it leads to a symmetric variational formulation. The Feng NRBCs [6] and the localized DtN conditions [8, 9] have this form.

Our goal is to rewrite (4) in a way that does not involve high-order derivatives. To this end, we introduce auxiliary variables defined on $\mathcal{B}$; i.e.,

$$v_0 = u, \tag{5}$$

$$v_j = v''_{j-1} = \partial_\theta^{2j} u, \quad j = 1, \ldots, K \quad \text{on } \mathcal{B}. \tag{6}$$

Here and elsewhere a prime indicates differentiation with respect to θ. The NRBC (4) together with these relations can be written in a matrix form as

$$\begin{Bmatrix} -\partial u/\partial r \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \dots & \alpha_{K-2} & \alpha_{K-1} \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix} \begin{Bmatrix} u \\ v_1 \\ v_2 \\ \vdots \\ v_{K-2} \\ v_{K-1} \end{Bmatrix} + \begin{bmatrix} 0 & 0 & \dots & 0 & 0 & \alpha_K \\ -1 & 0 & \dots & 0 & 0 & 0 \\ 0 & -1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & -1 & 0 & 0 \\ 0 & 0 & \dots & 0 & -1 & 0 \end{bmatrix} \begin{Bmatrix} u \\ v_1 \\ v_2 \\ \vdots \\ v_{K-2} \\ v_{K-1} \end{Bmatrix}'' . \tag{7}$$

This is a linear system of K equations and unknowns (on $\mathcal{B}$). To write this system more concisely, we define the vector of auxiliary variables as

$$\boldsymbol{U}^T = \{u \quad v_1 \quad v_2 \quad \cdots \quad v_{K-1}\}, \tag{8}$$

where the superscript T indicates transposition. We also let $\boldsymbol{e}_1$ be a vector (of dimension K) whose first entry is one and all other entries are zero. Thus (7) can be written as

$$-\frac{\partial u}{\partial r}\boldsymbol{e}_1 = \boldsymbol{A}\boldsymbol{U} + \boldsymbol{B}\boldsymbol{U}'' \quad \text{on } \mathcal{B}. \tag{9}$$

Here, $\boldsymbol{A}$ and $\boldsymbol{B}$ are the matrices appearing in (7). Since (9) involves only second-order derivatives, it is an AHOC according to our definition. However, (9) is not a desirable AHOC because the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are *nonsymmetric*, and therefore it would lead to a nonsymmetric FE formulation. The question is, thus, whether it is possible to replace (9) by the equivalent AHOC

$$-\frac{\partial u}{\partial r}\,\boldsymbol{e}_1 = \boldsymbol{Y}\boldsymbol{U} + \boldsymbol{Z}\boldsymbol{U}'' \quad \text{on } \mathcal{B}, \tag{10}$$

where the matrices $\boldsymbol{Y}$ and $\boldsymbol{Z}$ are *symmetric*. The answer is positive. In fact, it can be shown, using some simple matrix-theoretical arguments, that there are *unique* symmetric matrices $\boldsymbol{Y}$ and $\boldsymbol{Z}$ such that (10) implies (9). (see the Appendix for the proof of the uniqueness of the construction). These matrices are

$$\boldsymbol{Y} = \begin{bmatrix} \alpha_0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & -\alpha_2 & -\alpha_3 & \dots & -\alpha_{K-2} & -\alpha_{K-1} & -\alpha_K \\ 0 & -\alpha_3 & -\alpha_4 & \dots & -\alpha_{K-1} & -\alpha_K & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & -\alpha_{K-2} & -\alpha_{K-1} & -\alpha_K & 0 & \dots & 0 \\ 0 & -\alpha_{K-1} & -\alpha_K & 0 & 0 & \dots & 0 \\ 0 & -\alpha_K & 0 & 0 & 0 & \dots & 0 \end{bmatrix}, \tag{11}$$

$$\boldsymbol{Z} = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \dots & \alpha_{K-2} & \alpha_{K-1} & \alpha_K \\ \alpha_2 & \alpha_3 & \alpha_4 & \dots & \alpha_{K-1} & \alpha_K & 0 \\ \alpha_3 & \alpha_4 & \alpha_5 & \dots & \alpha_K & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{K-2} & \alpha_{K-1} & \alpha_K & 0 & 0 & \dots & 0 \\ \alpha_{K-1} & \alpha_K & 0 & 0 & 0 & \dots & 0 \\ \alpha_K & 0 & 0 & 0 & 0 & \dots & 0 \end{bmatrix}. \tag{12}$$

It is easy to verify that the symmetric linear system (10) with the matrices $\boldsymbol{Y}$ and $\boldsymbol{Z}$ given by (11) and (12), is indeed equivalent to the original linear system (7). In fact, the equations of the former are simply linear combinations of the equations of the latter. Equation (10) is the desired symmetric AHOC.

2.2. *NRBCs Involving Even and Odd Tangential Derivatives*

Now suppose that (4) is replaced by

$$-\frac{\partial u}{\partial r} = \sum_{j=0}^{K} \alpha_j \partial_\theta^j u \quad \text{on } \mathcal{B}. \tag{13}$$

In other words, suppose the initial sequence of NRBCs involves both even and odd tangential derivatives. Such an NRBC cannot lead to a symmetric FE formulation, but it can still be treated analogously to the previous case. In fact, all the equations of Section 2.1 are still valid, except that the second-order tangential derivative is replaced by a first-order derivative everywhere. Thus, the auxiliary variables are defined as

$$v_0 = u, \tag{14}$$

$$v_j = v'_{j-1} = \partial_\theta^j u, \quad j = 1, \dots, K \quad \text{on } \mathcal{B}, \tag{15}$$

and the symmetric AHOC is (cf. (10))

$$-\frac{\partial u}{\partial r}\boldsymbol{e}_1 = \boldsymbol{Y}\boldsymbol{U} + \boldsymbol{Z}\boldsymbol{U}' \quad \text{on } \mathcal{B}. \tag{16}$$

The matrices $\boldsymbol{Y}$ and $\boldsymbol{Z}$ are the same as in (11) and (12).

2.3. *NRBCs Involving Radial Derivatives*

Now we consider the case where only *radial* derivatives appear in the initial sequence of NRBCs. Such NRBCs have the form

$$-\frac{\partial u}{\partial r} = \alpha_0 u + \sum_{j=2}^{K} \alpha_j \partial_r^j u \quad \text{on } \mathcal{B}. \tag{17}$$

The Bayliss–Turkel [5] and the Higdon [7] NRBCs can be written in this form.

There are two ways to construct a symmetric AHOC for (17). The first is to mimic what has been done in the case of tangential derivatives. Thus, the auxiliary variables are defined similarly to (14) and (15), but the derivatives are now radial:

$$v_0 = u, \tag{18}$$

$$v_j = \partial_r v_{j-1} = \partial_r^j u, \quad j = 1, \ldots, K \quad \text{on } \mathcal{B}. \tag{19}$$

Consequently, the symmetric AHOC is (cf. (16))

$$-\frac{\partial u}{\partial r}\boldsymbol{e}_1 = \boldsymbol{YU} + \boldsymbol{Z}\partial_r \boldsymbol{U} \quad \text{on } \mathcal{B}. \tag{20}$$

Another difference from the previous case is that here $\alpha_1 = 0$ (from (17)). The matrices $\boldsymbol{Y}$ and $\boldsymbol{Z}$ in (20) are again the same as in (11) and (12), with $Z_{11} = 0$ in the present case.

The second way to obtain a symmetric AHOC for the NRBC (17) is to first replace the radial derivatives by tangential derivatives and then to construct the symmetric AHOC associated with the resulting NRBC in the manner described in Section 2.1. The replacement of r-derivatives by θ-derivatives is done by the recursive use of the Helmholtz equation (1), which is assumed to hold along $\mathcal{B}$ and inside Ω. Now we give the details of how this is done.

In polar coordinates, (1) becomes

$$\partial_r^2 u + \frac{1}{r}\partial_r u + \frac{1}{r^2}\partial_\theta^2 + \kappa^2 u = 0. \tag{21}$$

Thus, we have

$$\partial_r^2 u = C_{00}^{(2)}(r)u + C_{10}^{(2)}(r)\partial_r u + C_{01}^{(2)}(r)\partial_\theta^2 u, \tag{22}$$

where $C_{00}^{(2)} = -\kappa^2$, $C_{10}^{(2)} = -1/r$, and $C_{01}^{(2)} = -1/(r^2)$. The explanation for the notation is as follows: $C_{mk}^{(j)}$ is the coefficient of $\partial_r^m \partial_\theta^{2k} u$ appearing in the expression for $\partial_r^j u$. The third radial derivative can be expressed in terms of tangential derivatives by differentiating (22) with respect to r and then eliminating $\partial_r^2 u$ by using (22) again. The end result has the form

$$\partial_r^3 u = C_{00}^{(3)} u + C_{10}^{(3)}(r)\partial_r u + C_{01}^{(3)}(r)\partial_\theta^2 u + C_{11}^{(3)}(r)\partial_r \partial_\theta^2 u, \tag{23}$$

where the coefficients $C_{mk}^{(3)}$ can be expressed in terms of the coefficients $C_{mk}^{(2)}$; i.e.,

$$C_{00}^{(3)} = \partial_r C_{00}^{(2)} + C_{10}^{(2)} C_{00}^{(2)}, \tag{24}$$

$$C_{10}^{(3)} = C_{00}^{(2)} + \partial_r C_{10}^{(2)} + \left(C_{10}^{(2)}\right)^2, \tag{25}$$

$$C_{01}^{(3)} = \partial_r C_{01}^{(2)} + C_{10}^{(2)} C_{01}^{(2)}, \tag{26}$$

$$C_{11}^{(3)} = C_{01}^{(2)}. \tag{27}$$

In general, the expression for the jth radial derivative (for $j \geq 2$) can be reduced to the form

$$\partial_r^j u = \left(\sum_{k=0}^{J_0} C_{0k}^{(j)} + \sum_{k=0}^{J_1} C_{1k}^{(j)} \partial_r \right) \partial_\theta^{2k} u, \tag{28}$$

where

$$J_0 = \begin{cases} j/2; & j \text{ even} \\ (j-1)/2; & j \text{ odd} \end{cases}, \qquad J_1 = \begin{cases} (j-2)/2; & j \text{ even} \\ (j-1)/2; & j \text{ odd} \end{cases}. \tag{29}$$

General recursive formulas for the coefficients $C_{mk}^{(j)}$ are (for $j \geq 3$)

$$C_{0k}^{(j)} = \left\langle C_{1(k-1)}^{(j-1)} C_{01}^{(2)} \right\rangle^{\langle 0 \rangle} + \left\langle \partial_r C_{0k}^{(j-1)} \right\rangle^{\langle 1 \rangle} + \left\langle C_{1k}^{(j-1)} C_{00}^{(2)} \right\rangle^{\langle 2 \rangle}, \tag{30}$$

$$C_{1k}^{(j)} = C_{0k}^{(j-1)} + \left\langle \partial_r C_{1k}^{(j-1)} + C_{1k}^{(j-1)} C_{10}^{(2)} \right\rangle^{\langle 3 \rangle} \tag{31}$$

Here the following rules hold with respect to the indicated terms:

Term $\langle 0 \rangle$ is omitted if $k = 0$.
Term $\langle 1 \rangle$ is omitted if $j = 2k$.
Term $\langle 2 \rangle$ is omitted if $k = J_0$.
Terms $\langle 3 \rangle$ are omitted if $j = 2k + 1$.

The formulas (28)–(31) enable us to express any high-order radial derivative in terms of high-order tangential derivatives and the first-order radial derivative.

Now, by substituting (28) into the original NRBC (17), we obtain

$$-\frac{\partial u}{\partial r} = \alpha_0 u + \sum_{j=2}^{K} \alpha_j \sum_{k=0}^{K_0} \left(C_{0k}^{(j)} + C_{1k}^{(j)} \partial_r \right) \partial_\theta^{2k} u \quad \text{on } \mathcal{B}, \tag{32}$$

where

$$K_0 = \begin{cases} K/2; & K \text{ is even}, \\ (K-1)/2; & K \text{ is odd}. \end{cases} \tag{33}$$

In deriving (32), we exploit the fact that $J_1 \leq J_0 \leq K_0$ for $j \leq K$ (see (29)) and define $C_{mk}^{(j)}$ to be zero for indices beyond their original ranges of definition. By exchanging the order of the sums in (32), we then obtain

$$-\frac{\partial u}{\partial r} = \alpha_0 u + \sum_{k=0}^{K_0} (\gamma_k + \delta_k \partial_r) \partial_\theta^{2k} u \quad \text{on } \mathcal{B}, \tag{34}$$

where

$$\gamma_k = \sum_{j=2}^{K} \alpha_j C_{0k}^{(j)}, \qquad \delta_k = \sum_{j=2}^{K} \alpha_j C_{1k}^{(j)}. \tag{35}$$

Finally, we denote

$$\beta_0 = (\alpha_0 + \gamma_0)/(1 + \delta_0), \qquad \beta_k = (\gamma_k + \delta_k \partial_r)/(1 + \delta_0). \tag{36}$$

Then (34) becomes

$$-\frac{\partial u}{\partial r} = \sum_{k=0}^{K_0} \beta_k \partial_\theta^{2k} u. \tag{37}$$

This NRBC has the familiar form (4), although here the β_k are not constants but first-order operators. The procedure outlined in Section 2.1 can be applied formally to (37) to yield the symmetric AHOC (10.) The corresponding matrices $\boldsymbol{Y}$ and $\boldsymbol{Z}$ are given by (11) and (12), where α_k is replaced by β_k and K is replaced by K_0. From (36) it is clear that we can decompose $\boldsymbol{Y}$ and $\boldsymbol{Z}$ as

$$\boldsymbol{Y} = \boldsymbol{Y}_0 + \boldsymbol{Y}_1 \partial_r, \qquad \boldsymbol{Z} = \boldsymbol{Z}_0 + \boldsymbol{Z}_1 \partial_r, \tag{38}$$

where $\boldsymbol{Y}_0$, $\boldsymbol{Y}_1$, $\boldsymbol{Z}_0$, and $\boldsymbol{Z}_1$ are constant symmetric matrices. As a consequence, (10) becomes

$$-\frac{\partial u}{\partial r}\boldsymbol{e}_1 = \boldsymbol{Y}_0 \boldsymbol{U} + \boldsymbol{Y}_1 \partial_r \boldsymbol{U} + \boldsymbol{Z}_0 \boldsymbol{U}'' + \boldsymbol{Z}_1 \partial_r \boldsymbol{U}'' \quad \text{on } \mathcal{B}, \tag{39}$$

which is the desired symmetric AHOC.

2.4. *NRBCs Involving Tangential and Radial Derivatives*

If the initial sequence of NRBCs involves both tangential and radial high-order derivatives, one can proceed in either of two ways:

1. Eliminate the radial derivatives appearing in the NRBC using the Helmholtz equation recursively, as we have done in Section 2.3. Then use the procedure of Sections 2.1 and 2.2 to obtain a symmetric AHOC of either the form (39) (if only even-order tangential derivatives appear in the original NRBC) or the form

$$-\frac{\partial u}{\partial r}\boldsymbol{e}_1 = \boldsymbol{Y}_0 \boldsymbol{U} + \boldsymbol{Y}_1 \partial_r \boldsymbol{U} + \boldsymbol{Z}_0 \boldsymbol{U}' + \boldsymbol{Z}_1 \partial_r \boldsymbol{U}' \quad \text{on } \mathcal{B}. \tag{40}$$

2. Define auxiliary variables associated with *both* radial and tangential derivatives and treat both types of derivatives according to the procedure of Section 2.1. The precise way to do this will be explained in Section 4. The resulting two-level symmetric AHOC is analogous to the one discussed in Section 4.1 for time-dependent problems (the time derivative ∂_t in Section 4.1 being analogous to the radial derivative ∂_r here).

3. FINITE ELEMENT FORMULATION—THE ELLIPTIC CASE

Now we present the FE formulation for the problem in Ω consisting of the Helmholtz equation (1), the boundary condition on the obstacle surface (2), and the symmetric AHOC (10) on the artificial boundary $\mathcal{B}$. The unknown is the vector $\boldsymbol{U}$ defined in (8); its entries are u in Ω and the auxiliary variables v_j on $\mathcal{B}$.

The weak form of the problem is find $u \in H^1(\Omega)$ such that

$$a(w, u) + b(w, u) = L(w), \tag{41}$$

for any $w \in H^1(\Omega)$, where

$$a(w, u) = \int_\Omega \nabla w \cdot \nabla u \, d\Omega - \int_\Omega w \kappa^2 u \, d\Omega, \tag{42}$$

$$b(w, u) = -\int_B w(\partial u/\partial r) \, d\mathcal{B} = \cdots ?, \tag{43}$$

$$L(w) = \int_{\Gamma} wh\, d\Gamma. \tag{44}$$

The expression for $b(w, u)$ in (43) is still to be determined from (10). Now, we define the weighting function vector as

$$\boldsymbol{W}^T = \{w \quad \tau_1 \quad \tau_2 \quad \cdots \quad \tau_{K-1}\}, \tag{45}$$

where the $\tau_j \in H^1(\mathcal{B})$ are the weighting functions associated with the auxiliary unknowns $v_j \in H^1(\mathcal{B})$. Thus we have

$$\begin{aligned} b(w, u) &= -\int_{B} \omega(\partial u/\partial r)\, d\mathcal{B} = -\int_{B} \boldsymbol{W} \cdot \boldsymbol{e}_1(\partial u/\partial r)\, d\mathcal{B} \\ &= \int_{B} \boldsymbol{W} \cdot (\boldsymbol{YU} + \boldsymbol{ZU}'')\, d\mathcal{B} = \int_{B} \boldsymbol{W} \cdot \boldsymbol{YU} d\mathcal{B} - \int_{B} \boldsymbol{W}' \cdot \boldsymbol{ZU}'\, d\mathcal{B}. \end{aligned} \tag{46}$$

The one before last equality follows from (10), and the last equality is obtained by integration by parts. Note that $b(\cdot, \cdot)$ is a *symmetric* bilinear form. Thus, the weak form (41) can be written in terms of $\boldsymbol{U}$ and $\boldsymbol{W}$ as find $\boldsymbol{U} \in H^1$ such that for all $\boldsymbol{W} \in H^1$ there holds

$$\hat{a}(\boldsymbol{W}, \boldsymbol{U}) + \hat{b}(\boldsymbol{W}, \boldsymbol{U}) = \hat{L}(\boldsymbol{W}), \tag{47}$$

where

$$\hat{a}(\boldsymbol{W}, \boldsymbol{U}) = \int_{\Omega} \boldsymbol{\nabla} W_1 \cdot \boldsymbol{\nabla} U_1\, d\Omega - \int_{\Omega} W_1 \kappa^2 U_1\, d\Omega, \tag{48}$$

$$\hat{b}(\boldsymbol{W}, \boldsymbol{U}) = \int_{B} \boldsymbol{W} \cdot \boldsymbol{YU}\, d\mathcal{B} - \int_{B} \boldsymbol{W}' \cdot \boldsymbol{ZU}'\, d\mathcal{B}, \tag{49}$$

$$\hat{L}(\boldsymbol{W}) = \int_{\Gamma} W_1 h\, d\Gamma. \tag{50}$$

The Galerkin FE method is used to find an approximate solution. In each element, the functions $\boldsymbol{W}(\boldsymbol{x})$ and $\boldsymbol{U}(\boldsymbol{x})$ are replaced by their finite-dimensional approximations

$$\boldsymbol{W}^h(\boldsymbol{x}) = \sum_{a=1}^{N_{en}} \boldsymbol{W}_a N_a(\boldsymbol{x}), \qquad \boldsymbol{U}^h(\boldsymbol{x}) = \sum_{a=1}^{N_{en}} \boldsymbol{d}_a N_a(\boldsymbol{x}), \tag{51}$$

where $N_a(\boldsymbol{x})$ is the element shape function associated with node a and N_{en} is the number of element nodes. Of course, similar expressions can also be written on the global level. Note that the same shape functions, N_a, are used in (51) for the variable u and for all of the variables v_j. *This is not a constraint of the method*; in fact different shape functions may be used for different variables without affecting the symmetry of the formulation, although usually there is no need to do this. These approximations lead to the following FE formulation:

$$(\bar{\boldsymbol{K}} + \tilde{\boldsymbol{K}})\boldsymbol{d} = \boldsymbol{F} \tag{52}$$

$$\bar{\boldsymbol{K}} = \mathcal{A}_{e=1}^{N_{el}} \bar{\boldsymbol{k}}^e, \quad \tilde{\boldsymbol{K}} = \mathcal{A}_{e=1}^{N_{el}} \tilde{\boldsymbol{k}}^e, \quad \boldsymbol{F} = \mathcal{A}_{e=1}^{N_{el}} \boldsymbol{f}^e, \tag{53}$$

$$\bar{\boldsymbol{k}}^e = \left[\bar{k}^e_{(ai)(bj)}\right], \quad \tilde{\boldsymbol{k}}^e = \left[\tilde{k}^e_{(ai)(bj)}\right], \quad \boldsymbol{f}^e = \left\{ f^e_{(ai)} \right\}, \tag{54}$$

$$\overline{k}^e_{(ai)(bj)} = \delta_{i1}\delta_{j1} \int_{\Omega^e} (\nabla N_a \cdot \nabla N_b - N_a \kappa^2 N_b)\, d\Omega, \tag{55}$$

$$\tilde{k}^e_{(ai)(bj)} = \int_{B^e} (N_a Y_{ij} N_b - N'_a Z_{ij} N'_b)\, d\mathcal{B}, \tag{56}$$

$$f^e_{(ai)} = \delta_{i1} \int_{\Gamma^e} N_a h \, d\Gamma. \tag{57}$$

Here (ai) is the index associated with node a and "degree of freedom" i (for $i = 1, \ldots, K$), and similarly for (bj). Also, N_{el} is the number of elements, $\mathcal{A}_{e=1}^{N_{el}}$ is the assembly operator, δ_{ij} is the Kronecker delta, and Ω^e, $\mathcal{B}^e$, and Γ^e denote, respectively, the part of Ω, $\mathcal{B}$, and Γ associated with element e. The FE formulation (52)–(57) is C^0 and *symmetric*, as desired.

The solution of (52) yields the vector $\boldsymbol{d}$ whose entries are the approximate nodal values of $\boldsymbol{U}$ (see (51)). These nodal values include values of u in the interior domain Ω as well as values of v_j, namely, tangential derivatives of u along $\mathcal{B}$. The latter may be of interest to the analyzer; if not they should simply be ignored.

4. SYMMETRIC ARBITRARILY HIGH-ORDER CONDITIONS FOR THE TIME-DEPENDENT CASE

Now we consider the time-dependent scalar wave equation governing in the plane outside an obstacle:

$$\ddot{u} = c^2 \nabla^2 u + f. \tag{58}$$

Here a dot indicates differentiation with respect to time, c is the wave speed, and f is a given function with local support which is strictly contained in the finite domain Ω. On the obstacle boundary Γ, a Neumann condition holds:

$$\frac{\partial u}{\partial \nu} = h \quad \text{on } \Gamma. \tag{59}$$

Initial conditions are given as well:

$$u = u_I, \qquad \dot{u} = v_I \quad \text{at } t = 0. \tag{60}$$

Here u_I and v_I are given functions with local support strictly contained in Ω. As before, we introduce a circular artificial boundary $\mathcal{B}$ with radius R which encloses Ω (see Fig. 1b). On $\mathcal{B}$, an NRBC is applied, which is assumed to have the form (3). In the present case, the operator L_K involves temporal and spatial derivatives.

4.1. *NRBCs Involving Temporal and Even-Order Tangential Derivatives*

We start with the case where the initial sequence of NRBCs involves temporal and evenorder tangential derivatives. This is a generalization of the case considered in Section 2.1. The (K, P)-order NRBC has the form

$$-\frac{\partial u}{\partial r} = \sum_{j=0}^{K} \sum_{l=0}^{P} \alpha_{jl} \partial_\theta^{2j} \partial_t^l u \quad \text{on } \mathcal{B}, \tag{61}$$

where the α_{jl} are real constants.

As before, the idea is to introduce appropriate auxiliary variables to reduce the order of the derivatives. The vector of variables $\boldsymbol{U}$ is of length KP and is defined as

$$\boldsymbol{U}^T = \{u \quad v_{10} \quad \cdots \quad v_{(K-1)0} \quad v_{01} \quad v_{11} \quad \cdots \quad v_{(K-1)1} \quad \cdots \quad v_{0(P-1)} \quad \cdots \quad v_{(K-1)(P-1)}\}. \tag{62}$$

Here

$$v_{mn} = \partial_\theta^{2m} \partial_t^n u \quad \text{on } \mathcal{B}. \tag{63}$$

Our goal is to replace the NRBC (61) and the relation (63) by an AHOC of the form

$$-\frac{\partial u}{\partial r}\boldsymbol{e}_1 = \boldsymbol{S}\boldsymbol{U} + \boldsymbol{R}\boldsymbol{U}'' + \boldsymbol{P}\dot{\boldsymbol{U}} + \boldsymbol{Q}\dot{\boldsymbol{U}}'', \tag{64}$$

where all the arrays are of dimension KP and all the four matrices are *symmetric*. Unlike the elliptic case, there are many ways to construct such matrices. We choose the construction that is obtained by treating both types of derivatives analogously to the treatment of the tangential derivatives in the elliptic case (Section 2.1). This is done in two steps. First we "freeze" the time derivatives (or pretend that ∂_t^l is a scalar constant) and reduce the order of the tangential derivatives as in Section 2.1. This yields matrices $\boldsymbol{Y}$ and $\boldsymbol{Z}$ (cf. (11) and (12)), whose entries involve time-derivative operators. Then we reduce the time derivatives in each matrix entry using again an analogous procedure. We omit the details of the derivation and present the end result.

The matrices $\boldsymbol{S}$, $\boldsymbol{R}$, $\boldsymbol{P}$, and $\boldsymbol{Q}$ are

$$\boldsymbol{S} = \begin{bmatrix} \boldsymbol{E}_0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & -\boldsymbol{E}_2 & -\boldsymbol{E}_3 & \dots & -\boldsymbol{E}_{P-2} & -\boldsymbol{E}_{P-1} & -\boldsymbol{E}_P \\ 0 & -\boldsymbol{E}_3 & -\boldsymbol{E}_4 & \dots & -\boldsymbol{E}_{P-1} & -\boldsymbol{E}_P & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & -\boldsymbol{E}_{P-2} & -\boldsymbol{E}_{P-1} & -\boldsymbol{E}_P & 0 & \dots & 0 \\ 0 & -\boldsymbol{E}_{P-1} & -\boldsymbol{E}_P & 0 & 0 & \dots & 0 \\ 0 & -\boldsymbol{E}_P & 0 & 0 & 0 & \dots & 0 \end{bmatrix}, \tag{65}$$

$$\boldsymbol{R} \text{ is like } \boldsymbol{S},\ \text{ but each block } \boldsymbol{E}_j \text{ is replaced by } \boldsymbol{F}_j, \tag{66}$$

$$\boldsymbol{P} = \begin{bmatrix} \boldsymbol{E}_1 & \boldsymbol{E}_2 & \boldsymbol{E}_3 & \dots & \boldsymbol{E}_{P-2} & \boldsymbol{E}_{P-1} & \boldsymbol{E}_P \\ \boldsymbol{E}_2 & \boldsymbol{E}_3 & \boldsymbol{E}_4 & \dots & \boldsymbol{E}_{P-1} & \boldsymbol{E}_P & 0 \\ \boldsymbol{E}_3 & \boldsymbol{E}_4 & \boldsymbol{E}_5 & \dots & \boldsymbol{E}_P & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \boldsymbol{E}_{P-2} & \boldsymbol{E}_{P-1} & \boldsymbol{E}_P & 0 & 0 & \dots & 0 \\ \boldsymbol{E}_{P-1} & \boldsymbol{E}_P & 0 & 0 & 0 & \dots & 0 \\ \boldsymbol{E}_P & 0 & 0 & 0 & 0 & \dots & 0 \end{bmatrix}, \tag{67}$$

$$\boldsymbol{Q} \text{ is like } \boldsymbol{P},\ \text{ but each block } \boldsymbol{E}_j \text{ is replaced by } \boldsymbol{F}_j. \tag{68}$$

Each of the blocks $\boldsymbol{E}_j$ and $\boldsymbol{F}_j$ appearing above is a $K \times K$ matrix. They are given by

$$\boldsymbol{E}_j = \begin{bmatrix} \alpha_{0j} & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & -\alpha_{2j} & -\alpha_{3j} & \dots & -\alpha_{(K-2)j} & -\alpha_{(K-1)j} & -\alpha_{Kj} \\ 0 & -\alpha_{3j} & -\alpha_{4j} & \dots & -\alpha_{(K-1)j} & -\alpha_{Kj} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & -\alpha_{(K-2)j} & -\alpha_{(K-1)j} & -\alpha_{Kj} & 0 & \dots & 0 \\ 0 & -\alpha_{(K-1)j} & -\alpha_{Kj} & 0 & 0 & \dots & 0 \\ 0 & -\alpha_{Kj} & 0 & 0 & 0 & \dots & 0 \end{bmatrix}, \tag{69}$$

$$\boldsymbol{F}_j = \begin{bmatrix} \alpha_{1j} & \alpha_{2j} & \alpha_{3j} & \dots & \alpha_{(K-2)j} & \alpha_{(K-1)j} & \alpha_{Kj} \\ \alpha_{2j} & \alpha_{3j} & \alpha_{4j} & \dots & \alpha_{(K-1)j} & \alpha_{Kj} & 0 \\ \alpha_{3j} & \alpha_{4j} & \alpha_{5j} & \dots & \alpha_{Kj} & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{(K-2)j} & \alpha_{(K-1)j} & \alpha_{Kj} & 0 & 0 & \dots & 0 \\ \alpha_{(K-1)j} & \alpha_{Kj} & 0 & 0 & 0 & \dots & 0 \\ \alpha_{Kj} & 0 & 0 & 0 & 0 & \dots & 0 \end{bmatrix}. \tag{70}$$

The *two-level* structure of this construction is clear. Note the analogy between *all* these matrices and the matrices $\boldsymbol{Y}$ and $\boldsymbol{Z}$ (cf. (11) and (12)) in the elliptic case.

4.2. *Other Cases*

If the initial sequence of NRBCs involves time derivatives and *both even and odd* tangential derivatives, namely,

$$-\frac{\partial u}{\partial r} = \sum_{j=0}^{K} \sum_{l=0}^{P} \alpha_{jl} \partial_\theta^j \partial_t^l u \quad \text{on } \mathcal{B}, \tag{71}$$

then, although the FE formulation cannot be symmetrized, we employ a procedure analogous to that of Section 4.1. This results in the symmetric AHOC

$$-\frac{\partial u}{\partial r} \boldsymbol{e}_1 = \boldsymbol{S}\boldsymbol{U} + \boldsymbol{R}\boldsymbol{U}' + \boldsymbol{P}\dot{\boldsymbol{U}} + \boldsymbol{Q}\dot{\boldsymbol{U}}', \tag{72}$$

which is the same as (64) except that the *first* tangential derivative appears here instead of the second. The matrices $\boldsymbol{S}$, $\boldsymbol{R}$, $\boldsymbol{P}$, and $\boldsymbol{Q}$ remain the same as in Section 4.1.

If the initial sequence of NRBCs involves time derivatives and *radial* derivatives, namely,

$$-\frac{\partial u}{\partial r} = \sum_{j=0}^{K} \sum_{l=0}^{P} \alpha_{jl} \partial_r^j \partial_t^l u \quad \text{on } \mathcal{B}, \tag{73}$$

then, as in Section 2.3, there are two ways to construct a symmetric AHOC. The first is to treat the r-derivatives as the θ-derivatives have been treated above. Then, analogously to

(72), the resulting symmetric AHOC is

$$-\frac{\partial u}{\partial r}\boldsymbol{e}_1 = \boldsymbol{S}\boldsymbol{U} + \boldsymbol{R}\partial_r\boldsymbol{U} + \boldsymbol{P}\dot{\boldsymbol{U}} + \boldsymbol{Q}\partial_r\dot{\boldsymbol{U}}, \tag{74}$$

with the same coefficient matrices as before. The second way is to use the wave equation (58) recursively to replace the radial derivatives by (even-order) tangential and time derivatives (the procedure being similar to the one discussed in Section 2.3). This reduces the given NRBC to the form (61), but with the α_{jl} involving the ∂_r operator. Hence the resulting symmetric AHOC is of the form

$$-\frac{\partial u}{\partial r}\boldsymbol{e}_1 = \boldsymbol{S}_{000}\boldsymbol{U} + \boldsymbol{S}_{001}\dot{\boldsymbol{U}} + \boldsymbol{S}_{010}\boldsymbol{U}'' + \boldsymbol{S}_{100}\partial_r\boldsymbol{U} + \boldsymbol{S}_{011}\dot{\boldsymbol{U}}'' + \boldsymbol{S}_{101}\partial_r\dot{\boldsymbol{U}}$$
$$+ \boldsymbol{S}_{110}\partial_r\boldsymbol{U}'' + \boldsymbol{S}_{111}\partial_r\dot{\boldsymbol{U}}''. \tag{75}$$

Finally, we consider the case where the initial sequence of NRBCs involves *all types of derivatives*—temporal, tangential, and radial:

$$-\frac{\partial u}{\partial r} = \sum_{j=0}^{K}\sum_{m=0}^{M}\sum_{l=0}^{P}\alpha_{jml}\partial_\theta^j\partial_r^m\partial_t^l u \quad \text{on } \mathcal{B}. \tag{76}$$

Thus there are again two avenues for symmetric AHOC construction. First, one may treat each type of derivative *separately* as in Section 2.1. This leads to a *three-level* extension of the two-level construction in Section 4.1 and yields a symmetric AHOC of the form

$$-\frac{\partial u}{\partial r}\boldsymbol{e}_1 = \boldsymbol{S}_{000}\boldsymbol{U} + \boldsymbol{S}_{001}\dot{\boldsymbol{U}} + \boldsymbol{S}_{010}\boldsymbol{U}' + \boldsymbol{S}_{100}\partial_r\boldsymbol{U} + \boldsymbol{S}_{011}\dot{\boldsymbol{U}}' + \boldsymbol{S}_{101}\partial_r\dot{\boldsymbol{U}}$$
$$+ \boldsymbol{S}_{110}\partial_r\boldsymbol{U}' + \boldsymbol{S}_{111}\partial_r\dot{\boldsymbol{U}}'. \tag{77}$$

Here, the matrices $\boldsymbol{S}_{jkl}$ are of dimension KMP. They have the same structure as the matrices $\boldsymbol{S}$ and $\boldsymbol{P}$ in (65) and (67). Each block in these matrices has again the same structure, which in turn contains smaller subblocks. Each such subblock has yet again the same structure but now containing scalars for entries, as in (69) and (70).

Second, one may eliminate the radial derivatives by using the wave equation (58) to reduce the given NRBC to the form (71), which in turn leads to the symmetric AHOC (72) but with coefficients involving the operator ∂_r. Written differently, this AHOC again has the form (77).

5. FINITE ELEMENT FORMULATION—THE TIME-DEPENDENT CASE

Now we present the semidiscrete FE formulation for the problem in Ω consisting of the wave equation (58), the boundary condition on the obstacle surface (59), the symmetric AHOC (64) on the artificial boundary $\mathcal{B}$, and the initial conditions (60).

As in the elliptic case, a weak form of the problem can be written in terms of the unknown vector $\boldsymbol{U}$ and the weighting vector $\boldsymbol{W}$ (cf. (47)). The problem is then discretized in space using the Galerkin FE method. In each element, the functions $\boldsymbol{W}(\boldsymbol{x})$ and $\boldsymbol{U}(\boldsymbol{x}, t)$ are replaced by their finite-dimensional approximations

$$\boldsymbol{W}^h(\boldsymbol{x}) = \sum_{a=1}^{N_{en}} \boldsymbol{W}_a N_a(\boldsymbol{x}), \qquad \boldsymbol{U}^h(\boldsymbol{x}, t) = \sum_{a=1}^{N_{en}} \boldsymbol{d}_a(t) N_a(\boldsymbol{x}). \tag{78}$$

As in the elliptic case, the fact that the same shape functions N_a are used here for the variable u and for all the variables v_j is a matter of choice and not a constraint of the method. These approximations lead to the following FE linear dynamic system:

$$\boldsymbol{M}\ddot{\boldsymbol{d}}(t) + \boldsymbol{C}\dot{\boldsymbol{d}}(t) + \boldsymbol{K}\boldsymbol{d}(t) = \boldsymbol{F}(t). \tag{79}$$

This system is accompanied by appropriate initial conditions. The vector of initial values is easily obtained: it depends solely on the given functions u_I and v_I (see (60)) since all the auxiliary variables v_{mn} are defined along $\mathcal{B}$ only and thus, according to our assumption, vanish identically at time $t = 0$. The dynamic system (79) may be solved by a standard time-integration method, such as one of the Newmark family of schemes.

The expressions for the matrices and vectors appearing in (79) are

$$\boldsymbol{M} = \mathcal{A}_{e=1}^{N_{el}} \boldsymbol{m}^e, \quad \boldsymbol{C} = \mathcal{A}_{e=1}^{N_{el}} \boldsymbol{c}^e, \quad \boldsymbol{K} = \mathcal{A}_{e=1}^{N_{el}} \boldsymbol{k}^e, \quad \boldsymbol{F} = \mathcal{A}_{e=1}^{N_{el}} \boldsymbol{f}^e, \tag{80}$$

$$\boldsymbol{m}^e = \left[m^e_{(ai)(bj)}\right], \quad \boldsymbol{c}^e = \left[c^e_{(ai)(bj)}\right], \quad \boldsymbol{k}^e = \left[k^e_{(ai)(bj)}\right], \quad \boldsymbol{f}^e = \left[f^e_{(ai)}\right], \tag{81}$$

$$m^e_{(ai)(bj)} = \delta_{i1}\delta_{j1} \int_{\Omega^e} N_a N_b \, d\Omega, \tag{82}$$

$$c^e_{(ai)(bj)} = c^2 \int_{B^e} (N_a P_{ij} N_b - N'_a Q_{ij} N'_b) \, d\mathcal{B}, \tag{83}$$

$$k^e_{(ai)(bj)} = \delta_{i1}\delta_{j1} c^2 \int_{\Omega^e} \nabla N_a \cdot \nabla N_b \, d\Omega + c^2 \int_{B^e} (N_a S_{ij} N_b - N'_a R_{ij} N'_b) \, d\mathcal{B}, \tag{84}$$

$$f^e_{(ai)} = \delta_{i1} \int_{\Omega^e} N_a f \, d\Omega + \delta_{i1} \int_{\Gamma^e} N_a h \, d\Gamma. \tag{85}$$

The matrices $\boldsymbol{S}$, $\boldsymbol{R}$, $\boldsymbol{P}$, $\boldsymbol{Q}$ are those given in (65)–(68). Note the *damping* term $\boldsymbol{C}\dot{\boldsymbol{d}}$ in (79), which originates only from the AHOC (64) on $\mathcal{B}$ (the original problem having no physical damping associated with it). Note also the symmetry of the element-level FE matrices $\boldsymbol{m}^e$, $\boldsymbol{c}^e$, and $\boldsymbol{k}^e$, which implies the symmetry of the global-level matrices $\boldsymbol{M}$, $\boldsymbol{C}$, and $\boldsymbol{K}$ in (79).

6. NUMERICAL EXPERIMENTS

Now we present the results of some numerical experiments for the case of *time-harmonic waves* (Sections 2 and 3), with the symmetric AHOCs on $\mathcal{B}$, which are obtained from the *localized DtN conditions* [8, 9]. The latter NRBCs have the form (4) and thus lead to the AHOC (10). Preliminary results are reported in [21]. See [8] and [9] for details on how the coefficients α_j in (4) (and (10)) are defined. These two papers reach the same expression for α_j in two different ways: in [8] a local NRBC is constructed which exactly annihilates the first K cylindrical modes of the reflected wave, while in [9] the local NRBC of order K is found which is closest to the exact DtN condition in the L_2 norm. The latter approach leads to a two-parameter family of NRBCs, and when the two parameters coincide one obtains the localized DtN conditions.

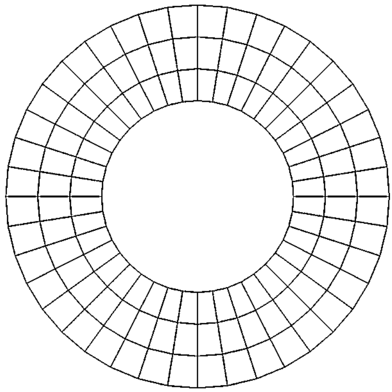

FIG. 2. Finite element mesh for the radiator problem.

In [8] these NRBCs were implemented and tested in their original form, namely, with high-order derivatives appearing explicitly in the FE formulation. To enable this, special FEs with high-order regularity were used in the layer adjacent to $\mathcal{B}$. This procedure was associated with two difficulties: (a) the NRBC order K could not be taken to be very large, because the programming of the special high-order FEs became too complex; (b) the even-order NRBCs turned out to be unstable. The latter fact has been theoretically verified in [8, 22]. Thus, only the odd-order localized DtN conditions are usable in their original form.

We consider a circular radiator of radius $a = 0.5$ in an infinite plane. Time-harmonic waves are propagated from the radiator's boundary Γ, with wave number $\kappa = 1$. On Γ we prescribe the values $\cos j\theta$, where j ranges from 0 (uniform radiation) to 5. We introduce a circular artificial boundary $\mathcal{B}$ of radius $R = 1$ around the radiator (see Fig. 1b). Thus the computational domain Ω is the annulus $a \leq r \leq R$. On $\mathcal{B}$ we apply the symmetric AHOC (10) which is obtained from the sequence of localized DtN conditions. Figure 2 shows the FE mesh, where bilinear quadrilateral elements are employed throughout. This means that bilinear shape functions are used for u in Ω and linear shape functions are used on $\mathcal{B}$ for all the auxiliary variables v_j.

Table I compares the exact solution with the FE solution obtained for different AHOC-orders K and for different radiation harmonics j. The value shown in all cases is the real part of the solution u at $r = R$ and $\theta = 0$. Naturally, all approximate solutions deteriorate when j becomes larger, since then the solution becomes more oscillatory while the mesh resolution remains the same. For a fixed value of j, the smallest error is obtained for $K = j$. Increasing K further does not improve the result. For $K \geq j$, the error associated with truncating the

TABLE I
Real Part of Solution u at $r = R$ and $\theta = 0$

j	Exact	$K = 1$	$K = 2$	$K = 3$	$K = 4$	$K = 5$
0	0.630	0.632	0.632	0.632	0.632	0.632
1	0.565	0.568	0.568	0.568	0.568	0.568
2	0.303	0.322	0.303	0.303	0.304	0.304
3	0.138	0.132	0.115	0.134	0.134	0.134
4	0.067	0.049	0.037	0.102	0.059	0.059
5	0.033	0.017	0.009	−0.019	0.010	0.026

TABLE II
Computational Parameters for Different NRBC Orders

Property	$K = 1$	$K = 2$	$K = 3$	$K = 4$	$K = 5$
Number of DOFs	120	160	200	240	280
Condition number	46.3	103.8	191.3	374.9	537.2
Relative CPU time	1.0	2.0	3.4	6.1	10.0

infinite domain is in fact zero. The error which remains is the FE discretization error and is not related to the NRBC. This error is about 0.3% for $j = 0$ (uniform radiation) and about 23% for $j = 5$.

Table II shows the total number of degrees of freedom, the condition number, and the relative CPU time for the different NRBC orders K. All of these computational parameters are monotonely increasing functions of K. Note the very low condition numbers obtained. In [8] , for the localized DtN conditions in their original high-derivative form, the condition numbers observed were orders of magnitude larger. Note also that $K = 5$ corresponds to an NRBC which involves a tangential derivative of order 10 (cf. (4)). With a standard FE formulation (i.e., with no auxiliary variables) this would require C^4 continuity along $\mathcal{B}$, which is very hard to achieve in practice. However, in the AHOC method, K can be increased very easily to any desired large value, since it is simply an input parameter.

One may wonder how the condition number grows as K becomes very large, e.g., with $K = 100$, which may be needed for extremely oscillatory solutions. The answer is that for such short waves a very fine mesh is needed, which would result in a high condition number regardless of the boundary condition used on $\mathcal{B}$. In fact, we have numerical evidence which shows that with fine resolution the density of the mesh is the dominant factor in determining the condition number, and that if we ignore the accuracy aspects and use a coarse mesh, the condition number for $K = 100$ would be about 50,000. This is still not regarded as a very large condition number. Thus, the boundary condition does not render the scheme more

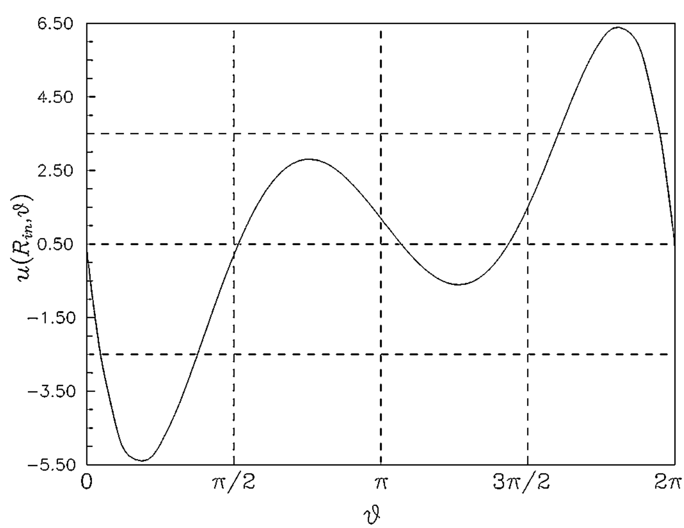

FIG. 3. Boundary condition on Γ. This is the fifth-order polynomial $g(\theta) = 0.4 - 24.02\theta + 31.25\theta^2 - 13.70\theta^3 + 2.47\theta^4 - 0.16\theta^5$ which has 2π-periodicity in both the function and its first derivative.

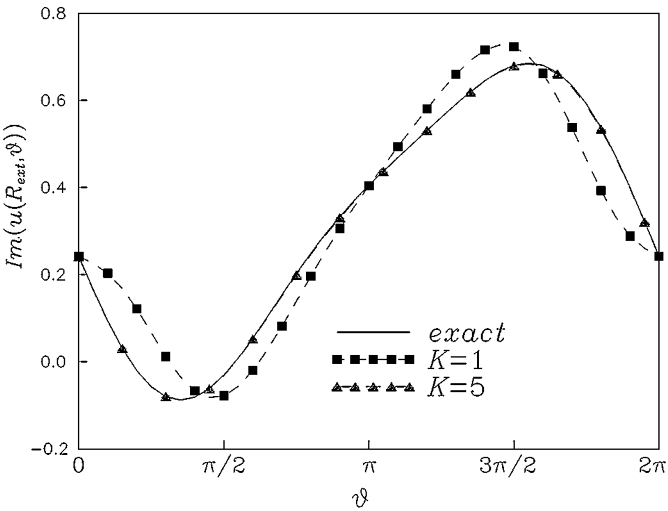

FIG. 4. Imaginary part of the solution on $\mathcal{B}$. Comparison of the exact solution, the first-order AHOC, and the fifth-order AHOC.

unstable than it already is due to the fine interior discretization. Detailed numerical results for problems with highly oscillatory solutions will be presented in a future publication.

The example just considered is degenerate in the sense that the exact solution is a single cylindrical mode. Now we consider another problem where the exact solution involves an infinite number of modes. To this end, we replace the boundary condition on the radiator surface Γ by the fifth-order periodic polynomial shown in Fig. 3. The imaginary part of the solution on $\mathcal{B}$ is shown in Fig. 4. Three solutions are compared in the figure: the exact solution and the numerical solutions obtained with the AHOCs of orders 1 and 5. As seen in the figure, the $K = 5$ solution is not distinguishable from the exact solution.

We observe from the numerical results that the AHOC is *stable for all orders* K. As mentioned previously, this is opposed to the situation occurring when the localized DtN conditions are used directly in the form (4) [8, 22]. Thus, in addition to all other advantages, the derivative-order reduction performed in the AHOC method has a stabilizing effect. This also demonstrates the known fact that one has to be careful when referring to the stability of a certain NRBC; stability is not a property of the NRBC alone, but a property of the NRBC, the method of its discretization, and the interior scheme combined.

7. CONCLUDING REMARKS

In this paper we have shown how to construct a local boundary condition of an arbitrarily high order with a symmetric structure, which is equivalent to a given high-order NRBC but does not involve any high-order derivatives. Such AHOCs, if incorporated in a numerical code, allow very easy accuracy control: to increase accuracy the user has only to increase the order of the AHOC which is simply an input parameter of the code. In this respect, symmetric AHOCs (which are local) are very similar to nonlocal NRBCs. Moreover, as discussed in the Introduction, if the truncation error associated with the AHOC of order K vanishes as $K \to \infty$, it is justified to think of the AHOC as *exact* just as the nonlocal DtN condition is regarded as exact.

We have numerically demonstrated the performance of one example of symmetric AHOCs, namely, those obtained from the localized DtN conditions, in the case of time-harmonic waves. In this case, the symmetric AHOCs lead to a symmetric C^0 FE scheme which is local, stable for all orders K, very accurate (as far as the truncation of the infinite domain is concerned), efficient, easy to implement, well-conditioned, and allows the use of standard element shape functions for all the variables. With regard to stability, it turns out in the case considered numerically that the symmetric AHOC, with low-order derivatives and auxiliary variables, is more stable than the original NRBC, with only one variable but with high-order derivatives.

Implementation of the AHOCs in the time-dependent case is under way. Results will be reported in a future publication. Theoretical stability analysis and error estimates for the AHOC form of the localized DtN conditions and of other sequences of NRBCs are also to be investigated.

APPENDIX: DERIVATION OF *Y* AND *Z* IN THE ELLIPTIC CASE

We consider the K-dimensional system of equations (7) or (9), i.e.,

$$-\frac{\partial u}{\partial r}\boldsymbol{e}_1 = \boldsymbol{A}\boldsymbol{U} + \boldsymbol{B}\boldsymbol{U}'' \quad \text{on } \mathcal{B}, \tag{A.1}$$

which constitutes a nonsymmetric AHOC. Here $\boldsymbol{e}_1$ is a K-vector whose first entry is one and all other entries are zero. We construct an equivalent AHOC of the form (10); i.e.,

$$-\frac{\partial u}{\partial r}\,\boldsymbol{e}_1 = \boldsymbol{Y}\boldsymbol{U} + \boldsymbol{Z}\boldsymbol{U}'' \quad \text{on } \mathcal{B}, \tag{A.2}$$

where the matrices $\boldsymbol{Y}$ and $\boldsymbol{Z}$ are symmetric.

To solve this problem we rephrase it as a problem in linear algebra: *Given the linear system of equations*

$$\boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{y} = s\boldsymbol{e}_1, \tag{A.3}$$

where s is a scalar and $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *are the matrices defined by (7), obtain, by applying elementary operations on the matrices, a new linear system of equations,*

$$\boldsymbol{Y}\boldsymbol{x} + \boldsymbol{Z}\boldsymbol{y} = s\boldsymbol{e}_1, \tag{A.4}$$

where $\boldsymbol{Y}$ *and* $\boldsymbol{Z}$ *are symmetric.* We shall show now how to derive $\boldsymbol{Y}$ and $\boldsymbol{Z}$, and in doing so we shall prove that the construction is unique.

First, we recall from linear algebra that a matrix $\boldsymbol{Y}$ is obtained from a matrix $\boldsymbol{A}$ by elementary operations iff $\boldsymbol{Y} = \boldsymbol{Q}\boldsymbol{A}$ for some nonsingular matrix $\boldsymbol{Q}$. Thus, we multiply the original system (A.3) by $\boldsymbol{Q}$ on the left, which yields

$$\boldsymbol{Q}\boldsymbol{A}\boldsymbol{x} + \boldsymbol{Q}\boldsymbol{B}\boldsymbol{y} = s\boldsymbol{Q}\boldsymbol{e}_1. \tag{A.5}$$

This should yield (A.4); hence $\boldsymbol{Y} = \boldsymbol{Q}\boldsymbol{A}$ and $\boldsymbol{Z} = \boldsymbol{Q}\boldsymbol{B}$ are symmetric, and we have $\boldsymbol{Q}\boldsymbol{e}_1 = \boldsymbol{e}_1$, which means that the first column of $\boldsymbol{Q}$ is $\boldsymbol{e}_1$. Now, we write the matrices $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{Q}$ in the following partitioned form:

$$\boldsymbol{A} = \begin{bmatrix} \alpha_0 & \boldsymbol{a}^T \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{b}^T \\ \boldsymbol{e} & \boldsymbol{F} \end{bmatrix}, \quad \boldsymbol{Q} = \begin{bmatrix} \boldsymbol{1} & \boldsymbol{q}^T \\ \boldsymbol{0} & \boldsymbol{T} \end{bmatrix}, \tag{A.6}$$

where

$$\boldsymbol{a}^T = \{\alpha_1 \quad \alpha_2 \quad \cdots \quad \alpha_{K-1}\}, \qquad \boldsymbol{b}^T = \{0 \quad 0 \quad \cdots \quad \alpha_K\}, \tag{A.7}$$

$$\boldsymbol{e} = -\boldsymbol{e}_1, \qquad \boldsymbol{F} = -\left[\boldsymbol{e}_2 \quad \boldsymbol{e}_3 \quad \cdots \quad \boldsymbol{e}_{K-1}\ \boldsymbol{0}\right]. \tag{A.8}$$

In (A.6)–(A.8), $\boldsymbol{I}$ is the $(K-1)\times(K-1)$ identity matrix, $\boldsymbol{e}_j$ is a vector with zero entries except a unit entry in the jth position, $\boldsymbol{q}$ is a $(K-1)$-vector, and $\boldsymbol{T}$ is a $(K-1)$-matrix. From this we calculate

$$\boldsymbol{Y} = \boldsymbol{QA} = \begin{bmatrix} \alpha_0 & \boldsymbol{a}^T + \boldsymbol{q}^T \\ \boldsymbol{0} & \boldsymbol{T} \end{bmatrix}, \qquad \boldsymbol{Z} = \boldsymbol{QB} = \begin{bmatrix} -q_1 & \boldsymbol{b}^T + \boldsymbol{q}^T\boldsymbol{F} \\ -\boldsymbol{Te}_1 & \boldsymbol{TF} \end{bmatrix}. \tag{A.9}$$

From the symmetry of $\boldsymbol{Y}$ we immediately deduce that $\boldsymbol{T}$ is symmetric and that $\boldsymbol{q} = -\boldsymbol{a}$. Using the latter equality as well as (A.7) and (A.8) in the expression for $\boldsymbol{Z}$ in (A.9) yields

$$\boldsymbol{Z} = \left[\begin{array}{cccc|c} \alpha_1 & \alpha_2 & \cdots & \alpha_{K-1} & \alpha_K \\ \hline & & & & 0 \\ & & -\boldsymbol{T} & & \vdots \\ & & & & 0 \end{array}\right]. \tag{A.10}$$

Thus, we have recovered the first row and first column in the matrix $\boldsymbol{Z}$ (cf. (12)). We now use the facts that $\boldsymbol{Z}$ given by (A.10) should be symmetric and that $\boldsymbol{T}$ itself should also be symmetric, and continue in this fashion recursively, to deduce finally that $\boldsymbol{Y}$ and $\boldsymbol{Z}$ must be the matrices given by (11) and (12). Since there was no freedom in the deduction process above, the construction is proved to be unique.

ACKNOWLEDGMENTS

This work was partly supported by the U.S.–Israel Binational Science Foundation and by the Fund for the Promotion of Research at the Technion. The author thanks Dr. Nomi Shaked-Monderer and Dr. Igor Patlashenko for their assistance.

REFERENCES

1. D. Givoli, *Numerical Methods for Problems in Infinite Domains* (Elsevier, Amsterdam, 1992).
2. S. V. Tsynkov, Numerical solution of problems on unbounded domains: A review, *Appl. Numer. Math.* **27**, 465 (1998).
3. D. Givoli, Exact representations on artificial interfaces and applications in mechanics, *Appl. Mech. Rev.* **52**, 333 (1999).
4. B. Engquist and A. Majda, Radiation boundary conditions for acoustic and elastic calculations, *Commun. Pure Appl. Math.* **32**, 313 (1979).
5. A. Bayliss and E. Turkel, Radiation boundary conditions for wave-like equations, *Commun. Pure Appl. Math.* **33**, 707 (1980).

6. K. Feng, Finite element method and natural boundary reduction, in *Proc. of the Int. Congress of Mathematicians, Warsaw*, 1983, pp. 1439–1453.
7. R. L. Higdon, Numerical absorbing boundary conditions for the wave equation, *Math. Comput.* **49**, 65 (1987).
8. D. Givoli, I. Patlashenko, and J. B. Keller, High-order boundary conditions and finite elements for infinite domains, *Comput. Meth. Appl. Mech. Eng.* **143**, 13 (1997).
9. D. Givoli and I. Patlashenko, Optimal local nonreflecting boundary conditions, *Appl. Numer. Math.* **27**, 367 (1998).
10. D. Givoli, Recent advances in the DtN FE method, *Arch. Comput. Meth. Eng.* **6**, 71 (1999).
11. D. Givoli and J. B. Keller, Special finite elements for use with high-order boundary conditions, *Comput. Meth. Appl. Mech. Eng.* **119**, 199 (1994).
12. I. Patlashenko and D. Givoli, Nonreflecting finite element schemes for three-dimensional acoustic waves, *J. Comput. Acoust.* **5**, 95 (1997).
13. F. Collino, High-order absorbing boundary conditions for wave propagation models. Straight line boundary and corner cases, in *Proc. 2nd Int. Conf. on Mathematical & Numerical Aspects of Wave Propagation*, edited by R. Kleinman *et al.* (SIAM, Delaware, 1993), pp. 161–171.
14. M. J. Grote and J. B. Keller, Exact nonreflecting boundary conditions for the time dependent wave equation, *SIAM J. Appl. Math.* **55**, 280 (1995).
15. M. J. Grote and J. B. Keller, Nonreflecting boundary conditions for time dependent scattering, *J. Comput. Phys.* **127**, 52 (1996).
16. M. J. Grote and J. B. Keller, Exact nonreflecting boundary condition for elastic waves, *SIAM J. Appl. Math.* **60**, 803 (2000).
17. T. Hagstrom and S. I. Hariharan, Progressive wave expansions and open boundary problems, in *Computational Wave Propagation*, edited by B. Engquist and G. A. Kriegsmann IMA Volumes in Mathematics and its Applications (Springer, New York, 1997), Vol. 86, pp. 23–43.
18. T. Hagstrom and S. I. Hariharan, A formulation of asymptotic and exact boundary conditions using local operators, *Appl. Numer. Math.* **27**, 403 (1998).
19. M. N. Guddati and J. L. Tassoulas, Transient analysis of wave propagation in unbounded media: Space-time methods and continued-fraction implementations, in *Computational Methods for Unbounded Domains*, edited by T. L. Geers (Kluwer, Dordrecht, 1998), pp. 169–178.
20. M. N. Guddati and J. L. Tassoulas, Continued-fraction absorbing boundary conditions for the wave equation, *J. Comput. Acoust.* **8**, 139 (2000).
21. D. Givoli and I. Patlashenko, An arbitrarily high-order local nonreflecting boundary scheme, in *Proc. ECCOMAS 2000*, edited by E. Onate, Barcelona, 2000.
22. A. Sidi and D. Givoli, Stability and accuracy of optimal local nonreflecting boundary conditions, *Appl. Numer. Math.* **33**, 327 (2000).